Experiment design for parameter estimation of a dynamical system in a nonlinear model structure based on multilevel excitation

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***Abstract*— An experiment design procedure for the estimation of the parameters of a dynamical system in a nonlinear model structure is presented in this paper. The input to the system is designed in such a way that the information content of the data, as measured by a scalar function of the information matrix is maximized. By restricting the input to a finite number of possible levels, the experiment design is formulated as a convex optimization problem which can be solved efficiently. The method is applied to a model of a Continuous Stirred Tank Reactor in a simulation study. The estimation based on the input signal obtained in our procedure is shown to outperform the one based on random binary signals.**

I. Introduction

The general problem considered in the Experiment Design (ED) field is to determine conditions such that the data collected from the experiments, together with the prior information available, allow us to construct an accurate model for the system.

In this paper, we restrict to the situation in which a *model structure* which can represent the true system is known (e.g. from first-principles knowledge), but the values of the *parameters* determining the "right" model within the structure are uncertain and need to be estimated. For dynamical systems, the model structure is a set of (ordinary, algebraic or partial) differential equation parametrized by the unknown parameters.

The objective of the ED in this case is to guarantee that accurate estimates for those parameters can be reconstructed from the data. A measure of the accuracy can be built by considering the so-called *information matrix*. A well-known result is indeed that when a statistically efficient method such as *maximum likelihood* is used for the parameter estimation, the variance of the estimated parameters is asymptotically equal to the inverse of the information matrix [21].

In the Systems Identification community, the ED task has been extensively studied for model structures of dynamical systems which are *linear* in the input [8]. In this case, the information matrix is an affine function of the *spectrum* of the input signal. This property has been widely used in the design of the excitation signal adopting a two-step design procedure. Firstly, a spectrum for the excitation signal which is optimal according to some criterion is determined. Exploiting the affine relation between the information matrix and the spectrum, the latter can often be found as the solution of a convex optimization problem [11], [2]. Secondly, a signal having the optimal spectrum is generated and given as input to the system.

Conversely, performing ED for parameter estimation of a dynamical system in a nonlinear model structure is still an open and challenging research topic. Still, the objective can be formulated in the terms of the information matrix as in the linear case. However, the spectrum of the input signal is not sufficient to characterize the information matrix for nonlinear dynamical system. A possibility is to design the entire probability density function of the input signal. [1] Since the probability density function appears linearly in the information matrix, a similar two-step procedure could be adopted. However, this procedure becomes much more involved than the one based on the spectrum used for linear dynamical system (see [10], [12], [20]).

An alternative approach is to optimize the input signal directly in the time domain by solving a dynamic optimization problem involving the information matrix [7]. This approach has been followed for different applications such as a cooling crystallization [4] and a semibatch RODTOX process [18]. However, a drawback is that the dynamic optimization problem in general very hard. Typically, it is severly nonconvex and depends on a large number of optimization variables representing a parametrization of the input signal. When the optimization problem is solved using standard gradient-based algorithms, chances are high that the numerical solution will lie in the proximity of a local optimum, which is possibly far away from the global one. Convex relaxation for ED problems posed in the time domain have only been developed for linear dynamical system (see [14]).

In this paper, we present an ED procedure which can be applied to a fairly large class of nonlinear model structures, but still relies on convex optimization. This method can be seen as a deterministic version of the approximate discrete design [16], [6] extended to dynamical systems.

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[1] Note that the spectrum (which is sufficient to characterize the information matrix for linear dynamical system) only describes the second order statistical properties of the input signal.

We restrict the range of the input signal to a finite number finite number of possible *levels* and divide the time of the experiment in a number N of consecutive intervals. During each of the intervals, we keep the input signal constant at one of the levels. This *piecewise constant* input signal can be described by the sequence of N levels that are encountered. We call this sequence the *input sequence*. In the input sequence we can recognize $N - m + 1$ shorter *subsequences* of length m. If the fading memory of the system is smaller than m intervals, then the output of the system during a certain interval j (and thus also the information matrix) is determined by the subsequence that ends in interval j. Therefore, the information matrix for the whole experiment is proportional to the *relative frequency* at which each subsequence occurs in the input sequence multiplied by the information matrix relative to the input subsequences. In other words, there is a linear relation between the information matrix and the relative frequencies.

Owing to linearity, we will be able to formulate the ED Problem optimizing a convex measure of the information matrix using the frequencies as design variables. After solving the problem, we generate an input sequence in which the subsequences appear in numbers proportional to the relative optimal frequencies.

Our approach is inspired by the *probabilistic input design* first proposed in [12] for nonlinear FIR systems. In that contribution, a convex measure of the expected value of the information matrix is optimized over the probability of the occurrence of the different subsequences. Compared to [12], we adapt the framework in such a way that it can be applied to the larger class of nonlinear *fading memory* systems. Moreover, we here follow a deterministic procedure for the generation of the input signal. For this reason, even though the ED optimization problem solved in [12] and the one solved here are formally equivalent, the probability of the subsequence is here interpreted as the frequency of occurrence in the input sequence.

Another similar ED method based on a multilevel input signal has been recently derived in [5]. Interestingly, the ED method in [5] was developed as a generalization of a previous multisine ED method for linear systems described in [17]. The ED method is described in [5] for the particular case of Wiener systems consisting of a linear FIR filter followed by a static polynomial nonlinearity. Compared to [5] (beside the more general framework allowing nonlinear fading memory systems), we formulate the ED as a convex optimization problem (as was also done in [12]) which can be solved efficiently using standard software and algorithms without introducing further approximations.

In the numerical example, we apply our method to a nonlinear model from process engineering (i.e. an irreversible CSTR reactor). We observe that the optimal input signal for our problem drives the system in two distinct regimes. This result is very interesting in relation to the classical ED problem involving linear systems, where it is known that the information matrix only depends on the spectrum of the input and therefore the latter can be generally chosen as the realization of an ergodic, quasi-stationary stochastic process. It is clear that a description in terms of the spectrum is not sufficient in order to characterize the optimal input signal in our case.

II. The Framework

A. True system and model structure

We assume that a *model structure* which can describe the true system is given in state-space ODE representation

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), \theta) \\ y(t) &= g(x(t), \theta) \end{aligned} \tag{1}$$

where $x(t)$ is the state, $u(t)$ is the input, $y(t)$ is the output and $\theta \in \mathbb{R}^p$ is the uncertain model parameter. The initial state $x_0 = x(0)$ is fixed and known. We consider here the SIMO case, i.e. $u(t), y(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}^q$ and $q \geq 1$. [2]

We assume that there exists one (and only one) *true parameter* θ^o such that the output of the model is equal to the output of the true system for every possible input signal.

Measurements $\tilde{y}_k$ of the output y are collected at a constant rate t_s and are corrupted by an additive white gaussian noise source e_k having mean 0 and covariance Σ_e: $\tilde{y}_k = y(kt_s) + e_k$ with $e_k \sim \mathcal{N}(0, \Sigma_e)$. We assume that the system has the *fading memory* property, in the sense defined in [3]. Loosely speaking, this means that the output of the system mostly depends on the values of the input in the recent past, while the influence of the input in the remote past gradually fades out.

B. Input signal

We restrict the range of the input signal $u(t)$ to a finite number of possible levels $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_{\ell-1}\}$. The time of the experiment is divided into a number of consecutive intervals $I_j,\ j = 0, 1, \ldots, N-1$. Each interval has a fixed duration t_I which is a multiple of the sampling time t_s: $t_I = nt_s$ where n is a positive, integer number. The input signal is kept constant to one of the possible levels in α during each interval.

The input signal during the experiment can be described by an input sequence of N levels $S = \{u_0,\ u_1, \ldots, u_{N-1}\},\ u_j \in \alpha$. Each element u_j represents the value of the input signal during the interval I_j. The output of the system $y(t)$ is a function of the input sequence S, the initial condition x_0 and the true parameter θ^o: $y(t) = y(t; S, x_0, \theta^o)$.

Since the system has fading memory, the output during a certain interval I_j mostly depends on the input during the previous m intervals $I_{j-m+1}, I_{j-m+2}, \ldots, I_j$. For this reason, we recognize in the input sequence S a number $N-m+1$ of shorter subsequences $s_j = \{u_{j-m+1}, u_{j-m+2}, \ldots, u_j\}$ having length m. Note that neighboring subsequences overlap on each other. In fact, two consecutive subsequences s_j and s_{j+1} share $m-1$ elements (see Figure 1).

[2] The extension to the MIMO case is possible, but would add notational complexity.

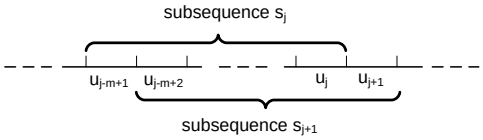

Fig. 1. Two consecutive subsequences s_j and s_{j+1} share $m-1$ elements.

The number m of elements in a subsequence is chosen large enough to describe the transient of the system with good accuracy. Thus, we can approximate the output of the system during interval I_j as a function of input values contained in the subsequence s_j, neglecting the influence of the input values outside s_j and of the initial condition[3], i.e. $y(t) \approx y(t; s_j, \theta^o)$ for $t \in I_j$ and $j \geq m-1$.

From a combinatorial consideration, the total number of possible subsequences is $M = \ell^m$. For the implementation of the ED Problem, it will be necessary to enumerate all the possible subsequences. To this end, it is convenient to put the set of the possible subsequences in a 1:1 relation with the set of the integer numbers ranging from 0 to $\ell^m - 1$. The relation is defined as follows. Given a subsequence, we consider its elements as the digits of an integer number written in the base ℓ. Conversely, given an integer number we derive its representation in the base ℓ and build the subsequence by appending the levels corresponding to the digits from left to right. We shall denote as $s[h]$ the subsequence which corresponds to the integer number h according to this criterion. For instance, the binary subsequence $\{\alpha_0, \alpha_1, \alpha_1\}$ corresponds to the number $(011)_2 = 3$. Conversely, if we deal with ternary subsequences the integer number $19 = (201)_3$ corresponds to the subsequence $\{\alpha_2, \alpha_0, \alpha_1\}$.[4]

C. Information Matrix

The information matrix F for the whole experiment is [5]

$$F = \sum_{j=0}^{N-1} \overbrace{\sum_{i=0}^{n-1} \psi((jn+i)t_s)\Sigma_e^{-1}\psi((jn+i)t_s)^\top}^{F_j}. \quad (2)$$

where $\psi(t) = \frac{d}{d\theta} y(t; S, x_0, \theta)\big|_{\theta=\theta^o}$ is the output parameter sensitivity.

We see that the information matrix F is the sum over the intervals of the information matrices F_j of the data contained in the intervals I_j. Due to the fading memory property of the system, the output of the system in the interval I_j (mostly) depends on the input values during the most recent m intervals. Therefore, F_j is a function of the subsequence $s_j = \{u_{j-m+1}, \ldots, u_j\}$ whose *last* element u_j appears in the input sequence S at the position j.[6] Furthermore, the information matrices F_l and F_m relative two the intervals l and m are identical if the subsequences s_l and s_m are the same.

For each of the possible subsequences, we define

$$p([h]) = \frac{\textit{number of occurrences of } s[h] \textit{ in } S}{N-m+1} \quad (3)$$

as its relative frequency in the input sequence S. [7] By grouping in (2) the identical terms and ignoring the contribution from data in the first $m-1$ intervals, the information matrix F can also be written as

$$F = (N-m+1) \sum_{h=0}^{M-1} p(s[h]) F(s[h]) \quad (4)$$

where $F(s[h])$ is the contribution to the information matrix relative to the subsequence $s[h]$.

In order to compute a term $F(s[h])$, we simply simulate the nonlinear system (together with the parameter sensitivities $\psi(t)$) feeding as input a signal which corresponds to $s[h]$. Following, we compute the information matrix $F(s[h])$ *for the last interval* of $s[h]$ as

$$F(s[h]) = \sum_{i=(m-1)n}^{mn-1} \psi(it_s)\Sigma_e^{-1}\psi(it_s)^\top \quad (5)$$

Note that the time in the latter equation is relative to the start of the subsequence $s[h]$.

III. Experiment Design

A. Relative frequencies as design variables

For a *given* input sequence S, it is probably more natural to compute the information matrix using (2). Nonetheless, (4) is useful for the *design* of the input sequence.

We see indeed that the information matrix F is a *linear function* of the relative frequencies $p(s[h])$. Thus, it is convenient to consider $p(s[h])$ as design parameters of the input sequence. Owing to the linear relation, a large class of ED problems can be posed as convex optimization problems in the decision variables $p(s[h])$.

However, a number of constraints on the relative frequencies have to be set in order to obtain a solution which can actually be implemented:

1) The relative frequencies $p(s[h])$ need to be positive numbers summing up to 1.
2) An *exact discrete design* [6] requires the relative frequencies $p(s[h])$ to be rational numbers in the range $\mathbb{Q}_N = \{\frac{k}{N},\ k = 1, 2, \ldots, N\}$.
3) The relative frequencies have to be chosen in such a way that the subsequences can be placed in the input sequence S as shown in Figure 1.

Addressing point 1) is immediate. It implies M linear inequalities and one linear equality, which can be easily implemented in a convex optimization problem.

[3]The approximation is not accurate for $j < m-1$, since the influence of the initial condition cannot be neglected for these intervals.

[4]The notation $(abc)_d$ here denotes the integer number having digits abc written in the base d. We omit to write the base when it is the usual base 10, e.g. $19 = (19)_{10}$.

[5]Note that due to chose framework (i.e. additive white noise on the output y) the information matrix F is a deterministic function of the input signal.

[6]We here assume that even the output parameter sensitivity $\psi(t)$ has the fading memory property.

[7]Note that we divide the number of occurrences of $s[h]$ in S by $N-m+1$ and not by N because the first $m-1$ elements of the sequence S do not contain any full subsequence on length m.

On the contrary, addressing point 2) is not straightforward. Restricting $p(s[h])$ to $\mathbb{Q}_N$ would lead to a hard combinatorial optimization problem. In practice, it is possible to relax (i.e. neglect) the constraint allowing $p(s[h])$ to be generic real numbers. In the ED literature, this corresponds to the so-called *continuous design* [6].

Point 3 is also delicate to address. We have seen that two consecutive subsequences s_j, s_{j+1} in the input sequence S have $m-1$ elements in common. Let $*$ denote any level $\alpha_h \in \alpha$. If the subsequence s_j is in the form $\{*, u_0, \dots, u_{m-2}\}$, the next subsequence s_{j+1} has to be in the form $\{u_0, u_1, \dots, u_{m-2}, *\}$. Similarly, a subsequence in the form $\{u_0, u_1, \dots, u_{m-2}, *\}$ can only come after by one the form $\{*, u_0, u_1, \dots, u_{m-2}\}$. Clearly, this sets a constraint on the number of subsequences which have these two patterns in the input sequence.

A *necessary* condition for the existence of an input sequence in which the subsequences appear in numbers proportional to the frequencies $p(s[h])$ is that

$$\sum_{h=0}^{\ell-1} p(\{\alpha_h, u_0, u_1, \dots, u_{m-2}\}) = \sum_{h=0}^{\ell-1} p(\{u_0, u_1, \dots, u_{m-2}, \alpha_h\}) \tag{6}$$

for all the ℓ^{m-1} possible of subsequences of length $m-1$ $\{u_0, u_1, \dots, u_{n-2}\}$, $u_j \in \alpha$. It is easy to show that the condition (6) is necessary, but not sufficient for the existence of an input sequence satisfying the ordering constraints. For instance, for a binary sequence of length 2, the solution $p(\{\alpha_0, \alpha_0\}) = p(\{\alpha_1, \alpha_1\}) = 0.5$, $p(\{\alpha_1, \alpha_0\}) = p(\{\alpha_0, \alpha_1\}) = 0$ satisfies the constraints (6). However, it is not possible to switch from the subsequence $(\{\alpha_0, \alpha_0\})$ to $(\{\alpha_1, \alpha_1\})$ without introducing at least one subsequence $\{\alpha_0, \alpha_1\}$. [8]

At the moment, we are not aware of the existence of a general, convex condition which is necessary and sufficient in order to satisfy the ordering constraints. Thus, we only consider the necessary condition (6) in the ED and solve the issues related to transitions similar to the one above by including a (minimum) number of additional elements in the input sequence.

Exploiting the correspondence between sequences and natural numbers, the subsequence $\{u_0, u_1, \dots, u_{m-2}\}$ corresponds to the integer number $(u_0 u_1 \dots u_{m-3} u_{m-2})_\ell$. Thus, the set of constraints (6) can be written conveniently as

$$\sum_{h=0}^{\ell-1} p(s[j+h\ell^{n-1}]) = \sum_{h=0}^{\ell-1} p(s[\ell j+h]) \text{ for } j = 0, 1, \dots \ell^{n-1}-1. \tag{7}$$

[8]The same issue is present in the approach presented in [5] where the same condition (6) has been used. A similar issue occurs following the probabilistic approach as in [12]. The condition (6) in the probabilistic approach guarantees the existence of a stationary distribution for the input sequence such that the subsequences have marginal probabilities $p(\{\cdot\})$. However, the Markov Chain defining this stationary probability distribution may not be *irreducible*. Therefore, it may not be possible to generate the subsequences in numbers asymptotically proportional to $p(s[h])$ sampling from a *single* realization in the Markov Chain (i.e. we may not be in the condition of applying the ergodic theorem [15]). These issues were not addressed in the contributions [12] and [5].

B. Experiment Design Problem

We here consider a D-optimal ED problem which aims to maximize the determinant of the information matrix. [9]

$$\max_{p(1), p(2), \dots, p(M)} \log\det\left(\sum_{h=0}^{M-1} p(s[h]) F(s[h])\right) \text{ subject to} \tag{8}$$

$$p(s[h]) \geq 0 \qquad \text{for } h = 0, 1, \dots, M-1 \tag{9}$$

$$\sum_{h=0}^{M-1} p(s[h]) = 1 \tag{10}$$

$$\sum_{h=0}^{\ell-1} p(s[j+h\ell^{n-1}]) = \sum_{h=0}^{\ell-1} p(s[\ell j+h]) \text{ for } j = 0, 1, \dots \ell^{n-1}-1 \tag{11}$$

The logarithm of the determinant of the information matrix (which is a *concave* function) is maximized (8) subject to constraints that the relative frequencies are positive (9) and sum up to 1 (10). The necessary ordering constraint (11) for the subsequences is also included.

The overall optimization problem (8)-(11) is convex and can be solved using standard software and algorithms. In this work, we used the optimization modelling software CVX [9] with the solver SDPT3 [19].

Remark 1: In order to solve the optimization problem (8)-(11), we need to compute the terms $F(s[h])$. However, these terms actually depend on the true parameters θ^o, which is in general unknown. This situation is common for ED problems and is known as the "chicken and the egg" issue. The most common workaround is to replace the true parameter θ^o by an initial estimate $\hat{\theta}$ in the ED problem.

Remark 2: Once the problem (8)-(11) is solved, an input sequence containing the subsequences in numbers (approximately) proportional to the optimal frequencies has to be generated. The input generation problem for dynamical systems is more complicated than in the static case considered in classical approximate discrete design due to the ordering constraints that the subsequences have to satisfy. In the numerical example, we will present an ad-hoc input generation for optimal frequencies obtained in that particular case. The development of a general algorithm for input generation starting from the optimal frequencies is left for future work.

IV. Numerical Example

We consider a Continuous Stirred Tank Reactor system with jacket cooling in which a first-order irreversible reaction $A \rightarrow B$ takes place. Details on this system can be found in Chapter 3 of [13]. An ODE representation of the system is

$$\dot{C}_A = \frac{F}{V_R}(C_{A0} - C_A) - C_A k_0 e^{-\frac{E}{RT}} \tag{12}$$

$$\dot{T}_R = \frac{F}{V_R}(T_0 - T_R) - \lambda \frac{C_A}{\rho c_p} k_0 e^{-\frac{E}{RT}} - \frac{U A_J (T_R - T_J)}{V_R \rho c_p} \tag{13}$$

$$\dot{T}_J = \frac{F_J}{V_J}(T_{cin} - T_J) + \frac{U A_J}{V_J \rho_J c_J}(T_R - T_J). \tag{14}$$

where C_A (kmol/m^3) is the concentration of the reactant A in the reactor, T_R (K) is the temperature inside the reactor

[9]Other ED problems considering a convex measure of the information matrix such as E-optimal, A-optimal and L-optimal [1] could be similarly implemented.

and T_J (K) is the temperature of the cooling medium inside the jackets. The input $u = F_J$ (m^3/s) is the flow of the cooling medium inside the jackets, the output vector is $y = [C_A\ T_R]^\top$ and the state vector is $x = [C_A\ T_R\ T_J]^\top$.

The symbols $\theta = [k_0\ E\ \lambda\ UA_J]^\top$ represent the uncertain parameters which are to be estimated. The other symbols represent known, fixed coefficients.

Measurements $\tilde{y} = [\tilde{C}_A\ \tilde{T}_R]^\top$ of $y = [C_A\ T_R]^\top$ are taken each $t_s = 10$ min. and are corrupted by additive white gaussian noise terms having variance $\sigma_C^2 = 0.05^2$ and $\sigma_T^2 = 0.1^2$ respectively: $\Sigma_e = \mathrm{diag}(\sigma_C^2, \sigma_T^2)$. The length of one interval is $t_I = 5$ hours, so one interval contains $n = 30$ time samples. An input subsequence is formed by $m = 10$ consecutive time intervals. A binary excitation signal is considered: $\alpha = \{\alpha_0,\ \alpha_1\}$ where $\alpha_0 = 0.6\bar{u}$, $\alpha_1 = 1.4\bar{u}$ and $\bar{u} = 11.26 \cdot 10^{-3}$. Thus, we have $M = 2^{10} = 1024$ possible input subsequences $s[h], h = 0, 1, \ldots, 1023$. For all the possible subsequences, the information matrix of the data collected in last element of the subsequence is computed.

The ED Problem (8-11) is implemented and solved numerically. In order to avoid the chicken and the egg issue, the ED is here based on the true parameters θ^o. [10]

Interestingly, out of the 1024 possible subsequencies $s[h]$ only 5 have a strictly positive optimal frequency $p(s[h])$:

$$p(\{\alpha_0,\alpha_0,\alpha_0,\alpha_0,\alpha_0,\alpha_0,\alpha_0,\alpha_0,\alpha_0,\alpha_0\}) = 0.1$$
$$p(\{\alpha_0,\alpha_1,\alpha_1,\alpha_1,\alpha_0,\alpha_1,\alpha_1,\alpha_1,\alpha_0,\alpha_1\}) = 0.225$$
$$p(\{\alpha_1,\alpha_0,\alpha_1,\alpha_1,\alpha_1,\alpha_0,\alpha_1,\alpha_1,\alpha_1,\alpha_0\}) = 0.225$$
$$p(\{\alpha_1,\alpha_1,\alpha_0,\alpha_1,\alpha_1,\alpha_1,\alpha_0,\alpha_1,\alpha_1,\alpha_1\}) = 0.225$$
$$p(\{\alpha_1,\alpha_1,\alpha_1,\alpha_0,\alpha_1,\alpha_1,\alpha_1,\alpha_0,\alpha_1,\alpha_1\}) = 0.225$$

We rename these subsequences to A, B, C, D, E respectively for notational convenience. It is easy to verify that the subsequences can be concatenated in two separated *cycles*: $L_1 = \{B, E, D, C\}$ and $L_2 = \{A\}$. Furthermore, we see that if we repeat the cycles L_1 and L_2 in a ratio 9:4, we obtain an input sequence in which the subsequences appear in a number proportional to the desired frequencies.

Therefore, we build an input sequence that contains exactly 9 cycles of L_1 followed by 4 cycle of L_2. The input sequence starts with a full subsequence B. Following, the last elements of the subsequences E, D, C, i.e. $\{\alpha_1, \alpha_1, \alpha_0\}$, are appended to the input sequence. In this way, the subsequences E, D, C are concatenated to the input sequence. The last elements of the subsequences B, E, D, C, i.e. $\{\alpha_1, \alpha_1, \alpha_1, \alpha_0\}$, are appended to the input sequence 8 more times, forming 9 loops L_1 in total. The input sequence defined so far terminates with one element α_0. Nine more elements α_0 are appended forming a full subsequence A, which is also a full cycle L_2. Three more elements α_0 are appended to the sequence, forming 4 cycles L_2 in total. The input signal $u^{\mathrm{oed}}(t)$ corresponding to this signal is formed by a square wave followed by a constant part at the value α_0 (Figure 2).

[10] Of course, this approach would not be feasible in practice since θ^o is always unknown. Nonetheless, the performance that we obtain using θ^o represents an upper bound to the performance that can be achieved adopting a different approximation and is interesting to consider this situation in this preliminary study.

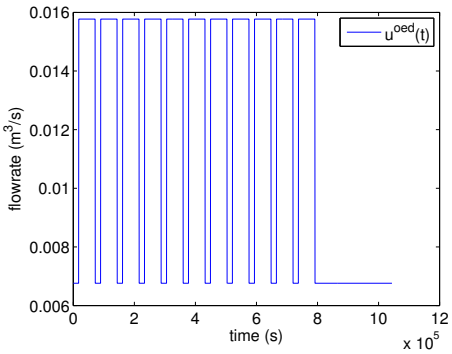

Fig. 2. Optimal input signal $u^{\mathrm{oed}}(t)$

In order to verify the effectiveness of the ED procedure, we perform two Monte Carlo studies where the parameter estimation is performed $n_{mc} = 100$ times for different realization of the measurement noise. The parameter estimation is performed according to the maximum likelihood criterion, which in this case is a weighted least squares criterion using the weights $\frac{1}{\sigma_C^2}, \frac{1}{\sigma_T^2}$ for the measurements C_A, T_R respectively.

In the first Monte Carlo study (Case 1), the input signal used is $u^{\mathrm{oed}}(t)$ for all the iterations. In the second Monte Carlo study (Case 2), the input $u_k^{rbs}(t)$ is a random binary signal defined between the same values and having clock period t_I. In this case, the input $u_k^{\mathrm{rbs}}(t)$ is stochastic and the realizations are different for all the Monte Carlo runs.

The sample-scaled determinant of the information matrices $\frac{1}{Nn} I_k^{\mathrm{oed}}$ and $\frac{1}{Nn} I_k^{\mathrm{rbs}}$ obtained for the different Monte Carlo iterations in the two cases are reported in Figure 3. Note that the information matrix in our case only depends on the input and therefore $I_k^{\mathrm{oed}} = I^{\mathrm{oed}}$ is the same for all the realizations in Case 1, while I_k^{rbs} depends on the particular realization of the random binary signal $u_k^{rbs}(t)$. The scaled determinant $\frac{1}{Nn} \det I^{\mathrm{oed}}$ is $2.2 \cdot 10^{14}$, while the average of the scaled determinants $\frac{1}{n_{mc}} \sum_{k=1}^{n_{mc}} \frac{1}{Nn} \det I_k^{\mathrm{rbs}}$ over the different Monte Carlo iterations is $1.2 \cdot 10^{14}$. Thus, in average, the optimal input is approximately 1.8 times more efficient than the random binary signal.

Under asymptotic assumptions, the covariance matrix of the estimated parameters equals the inverse of the information matrix. Nonetheless, it is interesting to check how accurate this relation is in practice. For this reason, the *sample covariance* $\hat{\Sigma}^{\mathrm{oed}} = \frac{1}{n_{mc}-1} \sum_{k=1}^{n_{mc}} (\hat{\theta}_k^{\mathrm{oed}} - \bar{\theta})(\hat{\theta}_k^{\mathrm{oed}} - \bar{\theta})^\top$ where $\bar{\theta} = \frac{1}{n_{mc}} \sum_{k=1}^{n_{mc}} = \hat{\theta}_k^{\mathrm{oed}}$ is compared with the *theoretical covariance* $\Sigma_{\mathrm{theo}}^{\mathrm{oed}} \triangleq \mathrm{inv}(I^{\mathrm{oed}})$. We find that

$$\hat{\Sigma}^{\mathrm{oed}} = \begin{bmatrix} 7.2 \cdot 10^{-2} & 9.5 \cdot 10^{-3} & 7.5 \cdot 10^{-4} & -4.2 \cdot 10^{-5} \\ 9.5e-3 & 1.3e-3 & 1.0e-4 & -1.0e-5 \\ 7.5e-4 & 1.0e-4 & 9.1e-5 & -4.5e-5 \\ -4.2e-5 & -1.0e-5 & -4.5e-5 & 2.6e-5 \end{bmatrix},$$
$$\Sigma_{\mathrm{theo}}^{\mathrm{oed}} = \begin{bmatrix} 8.6 \cdot 10^{-2} & 1.1 \cdot 10^{-2} & 5.7 \cdot 10^{-4} & 1.0 \cdot 10^{-4} \\ 1.1 \cdot 10^{-2} & 1.4 \cdot 10^{-3} & 8.e \cdot 10^{-4} & 1.0 \cdot 10^{-5} \\ 5.7 \cdot 10^{-4} & 8.0 \cdot 10^{-5} & 7.3 \cdot 10^{-5} & -3.5 \cdot 10^{-5} \\ 1.0 \cdot 10^{-4} & 1.0 \cdot 10^{-5} & -3.5 \cdot 10^{-5} & 2.0 \cdot 10^{-5} \end{bmatrix}.$$

The matrices $\hat{\Sigma}^{\mathrm{oed}}$ and $\Sigma_{\mathrm{theo}}^{\mathrm{oed}}$ are reasonably close to each other, e.g. the relative difference $\frac{\|\Sigma_{\mathrm{theo}}^{\mathrm{oed}} - \hat{\Sigma}^{\mathrm{oed}}\|_{\mathcal{F}}}{\|\Sigma^{\mathrm{oed}}\|_{\mathcal{F}}}$ of their Frobenius norm is 0.15.

Finally, a scatter plot for the first two coordinates of the

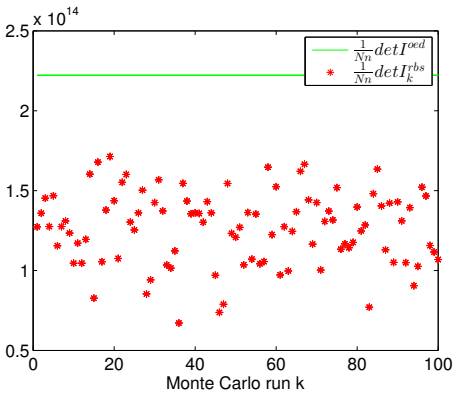

Fig. 3. Determinant of I_k^{oed} and I_k^{rbs} vs. iteration number k.

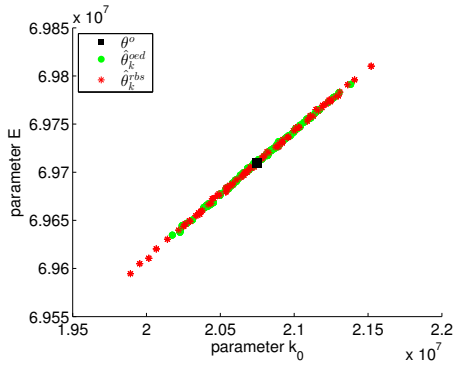

Fig. 4. Scatter plot for the first two coordinates of $\theta^o, \hat{\theta}_k^{\mathrm{oed}}$ and $\hat{\theta}_k^{\mathrm{rbs}}$

n_{mc} identified parameters for the two cases is reported in Figure 4. The parameters $\hat{\theta}_k^{\mathrm{oed}}$ identified in Case 1 are closer to the true parameter than the parameters $\hat{\theta}_k^{\mathrm{rbs}}$ identified in Case 2, as expected.

V. Conclusions

We have presented an ED method which can be applied to a wide class of nonlinear process model structures. The signal generated through this method is shown to compare favorably with random binary signals in a simulation case.

The promising results presented in this paper leave a number of open questions and space for future research. First, we would like to develop a general algorithm for the generation of the input sequence starting from the optimal frequencies. In general, this could be rather complicated when several cycles of subsequences are possible and when the transition between the cycles requires appending additional elements to the sequence. Tools from graph theory could be used to tackle the problem, e.g. to identify all the possible cycles between the subsequences.

Second, the complexity of our method is proportional to the number of possible subsequences, which in turn is equal to the number of levels raised to the power of the number of elements per subsequence. Increasing the number of levels allows one to switch between a larger number of input values. Increasing the number of elements in a subsequence allows one to switch more often between the input values. Both choices increase the degrees of freedom in the design and thus possibly lead to a more effective excitation signal. However, given the limitations on the computational power available a trade-off between considering more levels or more elements per subsequence has to be found.

Finally, in the numerical example we have observed that out of the many possible subsequences, only a few of them have a strictly positive optimal frequency, and they correspond to two different regimes for the system. We have obtained similar results for other nonlinear ED problems and we would like to investigate the reasons that lead to this.

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